

Symmetry and the Crossing Number for Complete Graphs*

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This paper studies the minimum number of intersections of edges in a complete graph on n vertices drawn in the plane. The proofs are first given for $n \leq 10$. A theorem on the maximum number of intersections is also given. Geometric representations of these cases are included. Symmetry of the representations is then discussed as it applies to extensions from small values of n to larger values maintaining the minimality of the number of intersections. Based on a symmetry conjecture given in the paper, a proof is given for the general case of the minimum intersection problem.

Key words: Complete graphs; intersections, symmetry.

1. Introduction

We are concerned with the conjecture that the minimum number of intersections I_n of the edges of a complete graph on n vertices drawn in the plane in which two edges intersect in at most one point and two edges with a vertex in common have no intersections, is given by

$$I_n = \frac{n(n-2)^2(n-4)}{64} \text{ for } n \text{ even}$$

$$I_n = \frac{(n-1)^2(n-3)^2}{64} \text{ for } n \text{ odd.}$$

In a complete graph every vertex is connected to each other vertex by one edge. We shall use C_n to denote a complete graph on n vertices and call it minimal (maximal) if it has a minimum (maximum) number of intersections. In reference [10]¹ a realization scheme to which we shall make frequent reference is given. Based on that construction and on those of related problems, the notion of symmetry for combinatorial problems is introduced and a conjecture in the form of a principle is given.

2. Proof of Small Cases

Consider each vertex of the graph C_n and its $n-1$ connecting edges (its star) to the remaining vertices. The removal of this vertex and its star, eliminates all intersections falling on these edges. The result is a complete graph on $n-1$ vertices. Any other vertex of the complete graph on n vertices has the same or a different number of intersections on its star. In general, if x_i , $i=1, \dots, n$ is the number of intersections on the edges connecting the i th vertex then the total number of intersections of C_n is $\frac{1}{4} \sum_{i=1}^n x_i$ since each intersection is counted exactly four times, once for each of the four vertices of the two edges defining the intersection. Thus we have:

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¹ Figures in brackets indicate the literature references at the end of this paper.

THEOREM 1: If $x_i, i=1, \dots, n$ is the number of intersections falling on the star of the i th vertex of a complete graph C_n , then the total number of intersections in the graph is given by $\frac{1}{4} \sum_{i=1}^n x_i$.

THEOREM 2: A necessary condition on the minimum number of intersections I_n of a complete graph C_n , is that it satisfy the relation

$$I_n \geq \frac{n}{n-4} I_{n-1} \quad n \geq 5$$

$$I_n = 0 \quad n \geq 4$$

PROOF: The average number of intersections per vertex in I_n is given by

$$\frac{4I_n}{n}$$

The removal of a vertex with at least an average number of intersections must leave a C_{n-1} whose intersection number is not less than I_{n-1} . Thus

$$I_n - \frac{4I_n}{n} \geq I_{n-1}$$

from which the relation follows.

Repetition of the above relation gives:

$$I_n \geq \frac{n(n-1)(n-2) \dots (n-k+1)}{(n-4)(n-5)(n-6) \dots (n-k-3)} I_{n-k}$$

from which lower bounds on I_n can be obtained if the minimal value of I_{n-k} is known. Guy [2] writes the above relation as:

$$I_n \geq I_t \binom{n}{t} \binom{n-4}{t-4}, 4 < t \leq n.$$

An inductive argument may be used to show that the conjectured formula for I_n holds for even n given that it holds for the immediately preceding odd case. This follows from

$$I_k - \frac{4}{k} I_k \geq I_{k-1}$$

$$I_k \geq \frac{k}{k-4} I_{k-1} = \frac{k}{k-4} \frac{(k-2)^2 (k-4)^2}{64} = \frac{k(k-2)^2 (k-4)}{64}.$$

However, if k is odd we have

$$I_k \geq \frac{k}{k-4} \left[\frac{(k-1)(k-3)^2(k-5)}{64} \right] = \frac{(k-1)(k-3)^2}{64} \left[\frac{k(k-5)}{k-4} \right]$$

$$= \frac{(k-1)(k-3)^2}{64} \left[(k-1) - \frac{4}{k-4} \right] = \frac{(k-1)^2(k-3)^2}{64} - 4 \frac{k-1}{k-4} \frac{(k-3)^2}{64}$$

and induction does not work from the even to the odd case.

THEOREM 3: $I_5 = 1$.

PROOF: (Kuratowski)[7].

THEOREM 4: $I_6 = 3$.

PROOF: Theorems 2 and 3 give $I_6 \geq 3$. However $I_6 = 3$ is realizable.

THEOREM 5: *Every complete graph on six vertices with a minimum number of intersections where its intersections are regarded as vertices, is isomorphic to the graph of the following figure:*

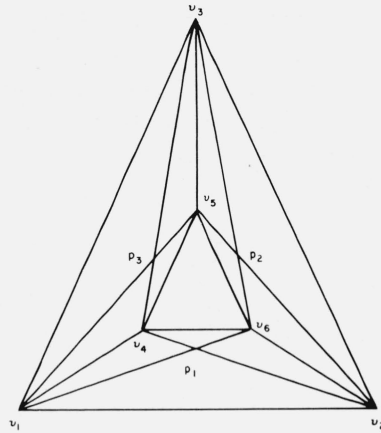


FIGURE 1

PROOF: Reference [9].

THEOREM 6: $I_7 = 9$, with unique representation for the corresponding C_7 to within stereographic projection.

PROOF: Theorem 2 may be applied repeatedly to $I_7 = 3, 4, 5, 6$ each time obtaining a vertex with such a number of intersections defined by its star, that its removal together with its star leaves a C_6 behind with less than 3 intersections, contradicting Theorem 4.

We now prove $I_7 \neq 8$ (the case $I_7 \neq 7$ is analogous). By Theorem 1 we have $\frac{8 \times 4}{7} > 4$. Thus there is at least one vertex with five intersections. Note that a vertex with 6 or more intersections would contradict Theorem 4. The only possibility here is to take for example $x_i = 5, i = 1, \dots, 4$ and $x_i = 4, i = 5, \dots, 7$. In any case the removal of a vertex which defines 5 intersections leaves a subgraph C_6 which is minimal. Using the figure of Theorem 5, we find that the 7th vertex may fall in any one of three different regions and connected to the rest of the graph. The other regions are symmetric with these. The three regions are: inside the interior triangle $v_4v_5v_6$ (outside the exterior triangle $v_1v_2v_3$ is symmetric with this region by stereographic projection); in a triangle such as $v_2v_6p_2$; and finally in a triangle such as $v_3v_6p_2$. In all three cases the edges connecting the seventh vertex to any of the remaining vertices must introduce at least six additional intersections, contradicting the fact that the vertex was assumed to introduce exactly five intersections. Thus $I_7 \geq 9$. However $I_7 = 9$ is realizable. Because C_7 with $I_7 = 9$ contains a C_6 with $I_6 = 3$ and this has a unique representation, our construction of C_7 is unique using symmetry arguments. See figure 5.

THEOREM 7: $I_8 = 18$.

PROOF: From Theorems 2 and 6 we have $I_8 \geq 18$. However $I_8 = 18$ is realizable. There is more than one representation.

THEOREM 8: $I_9 = 36$.

PROOF: By Theorem 2, $I_9 \geq 33$. Let $I_9 = x$. Since $\frac{18 \times 4}{8} = \frac{72}{8} = 9$ and $\frac{19 \times 4}{8} = \frac{76}{8} > 9$ we have for all three cases $x = 33, 34$, and 35 that the corresponding C_9 contains a minimal intersection C_7 .

Because a minimal intersection C_7 contains a minimal intersection C_5 and a minimal intersection C_3 , we have a contradiction to theorem 2 reference [10]. Since $x = 36$ is realizable, $I_9 = 36$. There is more than one representation.

THEOREM 9: $I_{10} = 60$.

PROOF: By Theorems 2 and 8, $I_{10} \geq 60$. However $I_{10} = 60$ is realizable. There is more than one representation.

3. Algebraic Formulation

The intersection problem for C_n may be algebraically formulated as follows:

Find nonnegative integers $x_i, i = 1, \dots, n$ such that (1) There is a geometric realization (drawing in the plane) of C_n with x_i intersections associated with the i th vertex and (2) these x_i yield a minimum integer value to

$$\frac{1}{4} \sum_{i=1}^n x_i$$

such that this realizable minimum I_n satisfies the condition

$$I_n \geq \frac{n}{(n-4)} I_{n-1}$$

given in Theorem 2.

We now prove that none of the components of the minimum solution vector $x = (x_1, \dots, x_n)$ is zero ($n \geq 6$). This is a corollary to:

THEOREM 10: *The maximum number of intersections of C_n is obtained by taking all its vertices on a polygon and joining them with edges in the interior of the polygon.*

PROOF: Since each intersection is determined by four vertices, $\binom{n}{4}$ is the maximum and the edges interior to the polygon give exactly this number of intersections since the diagonals of every quadrilateral intersect in the interior.

COROLLARY: *In a C_n $n \geq 6$ with a minimum number of intersections, the edges connecting any vertex to the remaining $n-1$ vertices must generate at least one intersection.*

PROOF: Otherwise all the vertices must be on a polygon and all their connecting lines must all be either inside or outside the polygon. However this gives a graph on $(n-1)$ vertices with a maximum number of intersections i.e., with $\binom{n-1}{4}$ intersections. But

$$\binom{n-1}{4} > \begin{cases} n(n-2)^2(n-4)/64 & n \text{ even} \\ (n-1)^2(n-3)^2/64 & n \text{ odd} \end{cases} \geq I_n$$

Thus at least one of the $n-1$ vertices must lie in a region defined by lines from the remaining $n-2$ vertices. In that case an edge from the n th vertex to the isolated vertex defines an intersection. This completes the proof.

General Considerations

Without the integer requirement on the value of $\frac{1}{4} \sum_{i=1}^n x_i$ or on its realizability, subject to the constraints we have:

THEOREM 11: *A necessary condition for a minimum (maximum) of $\frac{1}{4} \sum_{i=1}^n x_i$ is that $x_i = x_j$ (or are as equal as possible in the sense of integer values) for all i and j .*

PROOF: Both minimum and maximum values are bounded above and below by positive quantities and hence $x=(x_1, \dots, x_n)$ ranges over the lattice points of a bounded domain of E^n . The function $\prod_{i=1}^n x_i$ has a minimum over this domain. If we denote this minimum value by G we have

$$\frac{n}{4} G^{1/n} = \frac{n}{4} \left(\prod_{i=1}^n x_i \right)^{1/n} \leq \frac{n}{4} \sum_{i=1}^n \frac{x_i}{n}$$

The right side which gives the intersection number attains its minimum when the equality holds. Equality holds if and only if all the x_i are equal. (See ref. [5] page 27.)

In the following relation:

$$\frac{n}{4} \frac{\sum x_i}{n} \leq n^2 / 4 \sum \frac{1}{x_i}$$

the right side attains a maximum, H , as x ranges over the lattice points. This gives

$$\frac{n}{4} \sum \frac{x_i}{n} \leq n^2 / 4 \sum \frac{1}{x_i} = H.$$

The left attains its maximum when equality holds i.e., if and only if $x_i = x_j$ for all i and j (loc. cit. p. 26).

When $x_i = x_j = x$ for all i and j two difficulties may occur. The first is that

$$\frac{1}{4} \sum_{i=1}^n x_i = \frac{nx}{4}$$

may not be an integer. If it is an integer, it may not be possible to realize a complete graph such that all its vertices have the same number of intersections.

REMARK: Since the same basic formula $\frac{1}{4} \sum_{i=1}^n x_i$ holds for the number of intersections, this technique of equalizing the intersections among the vertices to obtain the minimum number would apply to any surface of genus p .

THEOREM 12: *A complete graph with a maximum number of intersections has a realization with $x_i = x_j$ for all i and j .*

PROOF: We have from the polygonal representation that

$$x_i = \frac{1}{4} \binom{n}{4} \quad i = 1, \dots, n$$

We also have as a consequence of the results of the early part of the paper:

THEOREM 13: *C_n for $0 \leq n \leq 10$, $n \neq 5, 7$ with a minimum number of intersections has a realization with $x_i = x_j$ for all i and j . For $n = 5, 7$, all but one vertex have the same number of intersections. The odd vertex has a number of intersections which differs by unity from those of the other vertices.*

PROOF: We only need to prove the assertion for the cases $n = 5$ and 7 . The other cases follow from the construction described in Theorem 1 reference [10]. The assertion for $n = 9$ can be seen to hold in the following figure in which each $x_i = 16$. The case $n = 5$ follows from $I_5 = 1$ and the fact that an intersection is determined by 4 vertices. The odd vertex has no intersections. For $n = 7$, since $I_7 = 9$ and 7 does not divide 36 not all x_i are equal. Thus $x_i = 5$ $i = 1, \dots, 6$ and $x_7 = 6$ see figure.

REMARK: In reference [1] a realization scheme which utilizes a Hamilton Circuit through the vertices with no intersections on its edges is described. This differs from the two concentric polygons scheme illustrated below (see figs. 6 and 8) and completely justified in reference [10].

We also have:

THEOREM 14: C_n for n even, whose intersection number is given by the conjecture and whose representation is given by the construction procedure of Theorem 1 reference [10] has a realization with $x_i = x_j$ for all i and j .

PROOF: It follows from Theorem 1 reference [10] that $x_i = x_j$ for all i and j .

4. Combinatorial Symmetry

This section is intended to give general combinatorial form to the conjecture at the beginning of the next section and, to some extent, to provide motivation for that conjecture. The reader may, however, omit this section and proceed directly to section 5.

In geometry, symmetry is defined by means of isometries called symmetry operations (and their groups of automorphisms) which leave a figure unchanged or invariant while permuting its parts [1a, 11]. The dimensions, translations and rotations, of the figure and its parts play an important role in characterizing its symmetry.

Here we introduce symmetry in a slightly more general sense. This symmetry depends on rules of construction applied to elements of a set appropriately partitioned into disjoint subsets together with a semigroup of maps between the members of the partition and lead to a particular combinatorial property that is preserved under the semigroup. Thus the symmetry operations leave the combinatorial property unchanged while the members of the partition of the set are interchanged elementwise.

An example is given by the representation of the maximum number of intersections of C_n . Here the set is partitioned into its singletons (the trivial partition), each member being a vertex. The representation is invariant to a transformation which carries a vertex and its star to another vertex and its star maintaining the order of the edges as they appear in the star. In general we can consider any partition.

We note in the following representation (fig. 2) of C_k $k = 4, 5, 6, \dots$ with a maximum number of intersections that the removal of any vertex and its star from C_n leaves a C_{n-1} with a maximum number of intersections. Here each C_n contains all lower order C_k 's $k = 1, 2, \dots, n$, each of which occurs $\binom{n}{k}$ times.

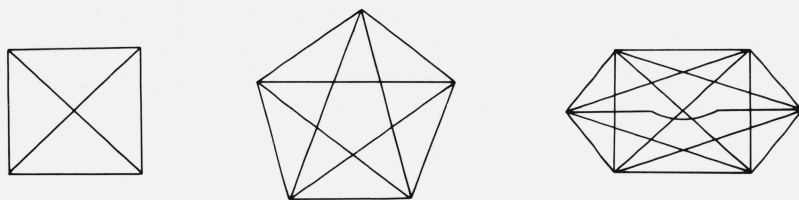


FIGURE 2

A Regeneration Principle of Combinatorial Symmetry

We find the following observation useful. It is that maximality is preserved when additional vertices are inserted in the figure having smaller number of vertices. Thus when symmetry is retained in the construction of a figure which is an extension of a combinatorially symmetric figure on a smaller number of elements and having a special character of the combinatorial property (e.g., maximality of intersections) then that special property is also preserved for the larger set. The process of extension may also be regarded as a superposition of a figure for a larger set of elements over the corresponding figure for a smaller set. In terms of the partition and the semigroup of maps, the insertion of vertices is an extension of the original figure, its partition and the semigroup of maps.

Thus the larger figure is an extension of a smaller figure with preservation of symmetry. The smaller one can then be obtained from the larger one by the removal of appropriate elements and yet smaller figures may be derived from the latter and so on. In this manner we obtain a descending sequence of figures each of which retains the relative combinatorial property. We refer to the process of extension with symmetry and preservation of relative properties as *regeneration*.

The possibility of symmetric regeneration suggests:

CONJECTURE (REGENERATION PRINCIPLE): *If a given combinatorial property is known to hold for the smallest order symmetric construction applied to a partition of a set and also hold for an appropriate subset of the family of its immediate ascendants (to insure that the rules of symmetric construction permit continued ascendance with preservation of the property) then the property holds for an arbitrary ascendant.*

Possibly the most essential criterion for developing symmetrization rules for minimal or maximal C_n is based on the following:

DEFINITION: *A symmetric realization of a complete graph is a symmetric drawing in the plane for which $\sum_{i,j=1}^n (x_i - x_j)^2 = \min$.*

Returning to the problem of minimal C_n , suppose that n is even. Symmetrization from C_{n-2} to C_n by introducing a pair of vertices must conform to the symmetry criteria already used to construct a symmetric C_{n-2} . Geometrically C_n is said to have the symmetric properties of C_{n-2} if its vertices can be divided into disjoint pairs with preservation of mutual relationship between pairs.

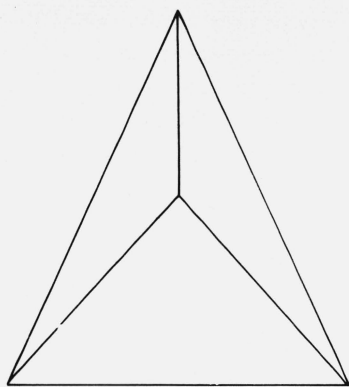
The pairs are interchangeable according to (1) the number of intersections sustained on their edges; (2) the rule or scheme according to which these edges are drawn. Thus a pair with its connecting edges (a double star) to the remaining C_{n-2} forms a configuration which is identical with that of any of the other $\frac{n}{2} - 1$ pairs. An edge in the double star of a pair has the same number of intersections following the rules of the symmetric construction as the corresponding edge in the double star of any other pair.

The removal of one vertex of any pair together with its star from C_n , n even, must yield the desired C_{n-1} with the conjectured number of intersections. To obtain C_{n-2} the other vertex of the pair must be removed.

The construction of a symmetric C_n for the conjectured value of I_n and satisfying the pairing requirement has been described in detail in the proof of Theorem 1 of reference [10]. The pairs are identified as in figure 8. One vertex belongs to the outer polygon and its companion is the farthest vertex (or one of the farthest if there is a tie) on the inner polygon.

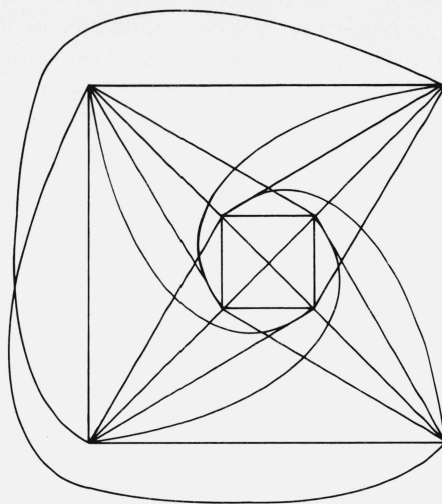
The removal of any one of the pairs of C_n naturally leaves a symmetric C_{n-2} . The addition of pairs to a lower order C_n must follow the rules of symmetry used to construct that C_n from its antecedent. If the Regeneration Principle holds then the graph is minimal. It can be shown to satisfy the conjectured value of I_n in 4 ways. (1) Two ways are given each by one of the two theorems of reference [10]; (2) since the pairs are chosen in the designated manner, it is trivial to note that all their corresponding C_4 's are minimal and hence the discussion on pages 574-575 of reference [10] applies; (3) see section 5.

This conjecture supports a counting method used below. A C_8 is the smallest complete subgraph of a C_n in which all three types of intersections which can arise between pairs occur. These are intersections (a) between two pairs, (b) between three pairs i.e., the intersecting edges have their end points among the three pairs and (c) between four pairs (each of the four end points belongs to a different pair).



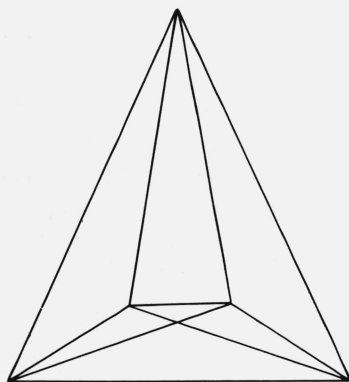
$$I_4 = 0$$

FIGURE 3



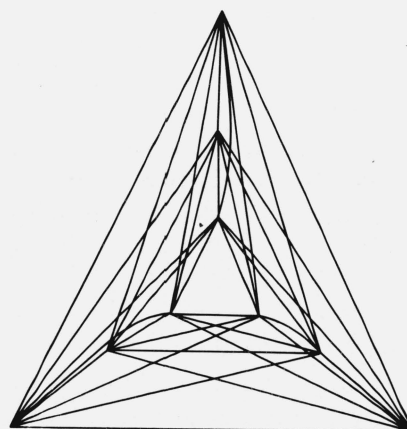
$$I_8 = 18$$

FIGURE 6



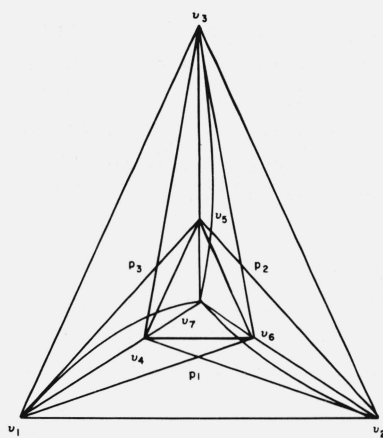
$$I_5 = 1$$

FIGURE 4



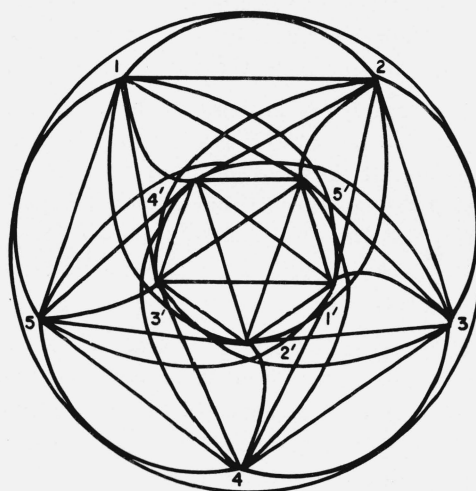
$$I_9 = 36$$

FIGURE 7



$$I_7 = 9$$

FIGURE 5



$$I_{10} = 60$$

FIGURE 8

5. Conjecture

If C_n has a minimal intersection representation from which one may choose $r = \left\lceil \frac{n}{2} \right\rceil$ pairs P_1, \dots, P_r of vertices of C_n such that: If C_{2s} is the full subgraph of C_n determined by $P_1 \cup \dots \cup P_s$ ($1 \leq s \leq r-1$), then C_{2s} is minimal. Moreover, if n is odd with vertices $v \in P_1 \cup \dots \cup P_r$ then any C_7 determined by vertices $v \cup P_i \cup P_j \cup P_k$, $i \neq j \neq k$, is minimal.

REMARK: Note that the Regeneration Principle implies the Conjecture.

THEOREM 15: If the above Conjecture is true (and hence if the Regeneration Principle is true), then I_n has the value conjectured.

PROOF BY INDUCTION ON n : We assume that I_m has the conjectured value for $m < n+2$ and assert that I_{n+2} does also. If n is even then $n+1$ is odd so I_{n+1} has the conjectured value and by the remark after theorem 2, I_{n+2} has the conjectured value. Therefore we suppose that n is odd say $n=2k+1$.

Let C_{n+2} be arbitrary and use the conjecture to choose pairs P_1, \dots, P_{k+1} of vertices with one vertex v left over. By induction $C_{n+1} = C_{n+2} - v$ has at least I_{n+1} intersections. Since $I_{n+2} - I_{n+1} = k \binom{k+1}{2}$, we must show that adding the vertex v to this C_{n+1} introduces at least $k \binom{k+1}{2}$ new intersections. Now by the conjecture the C_{n-1} determined by $P_1 \cup \dots \cup P_k$ is minimal so the addition of v to this C_{n-1} must introduce at least $(k-1) \binom{k}{2}$ intersections. Thus, we need only show that

$$k \binom{k+1}{2} - (k-1) \binom{k}{2} = 3 \binom{k}{2} + k$$

new intersections are introduced by the addition of v to the C_{n+1} determined by $P_1 \cup \dots \cup P_{k+1}$ but not merely by the addition of v to the C_{n-1} determined by $P_1 \cup \dots \cup P_k$.

To see that we can do this, consider any pair of pairs P_i, P_j , $1 \leq i \neq j \leq k$. By the conjecture $v \cup P_i \cup P_j \cup P_{k+1}$ determine a minimal C_7 with nine intersections. It follows from theorem 6 that exactly three of these intersections involve v and all three of the pairs P_i, P_j, P_{k+1} . Thus we obtain $3 \binom{k}{2}$ new intersections as $\{i, j\}$ ranges over $\{1, \dots, k\}$. We obtain at least k more intersections by taking all C_5 's of the form $P_i P_{k+1} v$. This completes the proof.

6. Generalization

The minimum intersection problem for C_n is a special case of the following general problem. Consider a connected graph G with n vertices joined by edges in the plane as follows: each vertex is joined to k other vertices ($2 \leq k \leq n-1$) in such a way that a maximum number of vertices has degree k . If any two edges can cross at most once at a point other than a vertex, determine the minimum number of intersections $I_n(k)$ of the edges of G .

The solution of this problem should follow from our previous discussion of symmetric realization. Our existing realization may be used to remove the necessary number of edges.

It is clear for example that the removal of the edges joining the prescribed pairs in the even case deletes the largest number of intersections. In the odd case one vertex must have no edge removed otherwise another vertex would have $(n-2)$ edges removed. The operation of removing an edge per vertex is continued. This gives

$$I_n(n-2) = \begin{cases} \frac{n(n-2)(n-4)(n-6)}{64} & n \text{ even} \\ \frac{(n-1)(n-3)^2(n-5)}{64} & n \text{ odd} \end{cases}$$

$$I_n(n-3) = \begin{array}{ll} \frac{n(n-4)(n-6)^2}{64} & n \text{ even} \\ \frac{(n-3)^2(n-5)^2}{64} & n \text{ odd} \end{array}$$

$$\begin{array}{l} \cdot \quad \cdot \quad \cdot \\ I_n(3) = 0 \\ I_n(2) = 0 \\ I_n(1) = 0 \\ I_n(0) = 0. \end{array}$$

Induction on k may be useful to go from the lower cases to higher ones.

REMARK: If the regeneration principle holds, then the proof of Zarankiewicz's problem for the minimum intersection number of edges joining every vertex of one set to every vertex of another set in a bipartite graph would follow from his symmetric construction.

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